

ON A LENGTH PRESERVING CURVE FLOW

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ABSTRACT. In this paper, we consider a new length preserving curve flow for convex curves in the plane. We show that the global flow exists, the area of the region bounded by the evolving curve is increasing, and the evolving curve converges to the circle in C^∞ topology as $t \rightarrow \infty$.

1. INTRODUCTION

In this paper, we study a nature evolution non-local flow for convex curves in the plane. This flow preserves the length of curves. We shall obtain the entropy estimate and integral estimates for the evolution to get a global flow. We remark that this kind of technique was used in [7], where they studied the curve shortening flow. Curve shortening flow has been studied extensively in the last few decades (see [7] and [12] for background and more references). It can be showed that the convexity of curves along the curve shortening flow is preserved and the curves become more and more circular before they collapse to a point. Since then, other flows for curves have also been proposed. One may see B.Andrews's papers (see for example, [1]) and Tsai's papers [17] for other kind of flows for curves. One may see [10] for higher dimension flows for hyper-surfaces. As showed by M.Gage [6], some non-local flow for convex curves are also very interesting. In a very recent paper [16], S.L.Pan and J.N.Yang consider a very interesting length preserving curve flow for convex curves in the plane of the form

$$\frac{\partial}{\partial t}\gamma(t) = \left(\frac{L}{2\pi} - k^{-1}\right)N,$$

where L , N , and k are the length, unit normal vector, and the curvature of the curve $\gamma(t)$ respectively. They have proved that the convex plane curve will become more and more circular and converges to circle in the C^∞ sense. It is interesting to study curve flow which preserve some geometry quantity, such as the area of the region bounded by the curve. For this, one may see [13] for a recent study.

The main result of this paper is the following theorem.

Theorem 1. *Suppose $\gamma(u, 0)$ is a convex curve in the plane R^2 . Assume $\gamma(t) := \gamma(u, t)$ satisfies the following evolving equation*

$$(1.1) \quad \frac{\partial}{\partial t}\gamma(t) = (k - \alpha(t))N,$$

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where k is the curvature of the curve $\gamma(t)$ and

$$\alpha(t) = \frac{1}{2\pi} \int k^2 ds.$$

Then the flow (1.1) is a length preserving flow. Furthermore, we have the global flow $\gamma(t)$ and $\gamma(t)$ converges in C^∞ to the circle of the fixed length L , as $t \rightarrow \infty$.

We remark that for $\alpha(t) = \frac{2\pi}{L}$, where L is the length of the curve $\gamma(t)$, the evolution equation (1.1) is area-preserving flow, which was studied by M.Gage in [6].

The paper is organized as follows.

In section 2, we introduce necessary formulae for the flow (1.1). We obtain key estimates about the curvature of the curve flow (1.1) in section 3. We can justify the assumption that the curve is a convex curve and the flow does not blow up in finite time. We obtain the theorem 1 in the C^0 case. In the last section, we show the C^∞ convergence of the flow.

2. PREPARATION

First of all, we derive basic formulae for our curve flow (1.1).

Lemma 2. *Let $w = |\gamma_u|$. Then we have*

$$w_t = -k(k - \alpha)w,$$

and

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} - \frac{\partial}{\partial s} \frac{\partial}{\partial t} = k(k - \alpha).$$

Proof. Note that

$$w^2 = |\gamma_u|^2.$$

Then we have

$$ww_t = \langle \gamma_u, \gamma_{tu} \rangle = \langle \gamma_u, ((k - \alpha)N)_u \rangle = w^2(k - \alpha) \langle T, N_s \rangle.$$

Using

$$N_s = -kT,$$

we have

$$w_t = -k(k - \alpha)w.$$

So,

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial}{\partial s} - \frac{\partial}{\partial s} \frac{\partial}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{1}{w} \right) \frac{\partial}{\partial u} \\ &= k(k - \alpha) \partial_s. \end{aligned}$$

□

Recall that $ds = wdu$. Then we have

$$(ds)_t = w_t du = -k(k - \alpha)ds.$$

We shall use this formula later.

Lemma 3.

$$\frac{\partial}{\partial t} T = \partial_s k N.$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}T &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\gamma = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\gamma + k(k-\alpha)\partial_s\gamma \\ &= \partial_s((k-\alpha)N) + k(k-\alpha)T \\ &= \partial_s k N.\end{aligned}$$

□

We denote the angle between the tangent and the X-axis by θ . Then we have

$$\cos\theta = \langle T, X \rangle$$

and

$$k = \frac{\partial\theta}{\partial s}.$$

Lemma 4.

$$\frac{\partial\theta}{\partial t} = \partial_s k.$$

Proof.

$$-\sin\theta \frac{\partial\theta}{\partial t} = \langle \frac{\partial T}{\partial t}, X \rangle = \partial_s k \langle N, X \rangle.$$

□

Then we can derive the important evolution equation for curvatures.

Lemma 5.

$$\frac{\partial}{\partial t}k = \partial_s^2 k + k^2(k-\alpha).$$

Proof.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\theta = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\theta + k(k-\alpha)\partial_s\theta = \partial_s^2 k + k^2(k-\alpha).$$

□

Hence, we have

Lemma 6.

$$\frac{\partial}{\partial t} \int k ds = 0.$$

Proof.

$$\frac{\partial}{\partial t} \int k ds = \int \frac{\partial}{\partial t} k ds + k \frac{\partial}{\partial t} ds = \int \partial_s^2 k + k^2(k-\alpha) - k^2(k-\alpha) ds = 0.$$

□

Using this, we now derive

Remark 7. Since $\gamma(0)$ is a convex curve in the plane, we have $\int k ds = 2\pi$.

Hence, we have

Lemma 8. *The length of the curve is fixed under the flow.*

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}L &= \int \frac{\langle \gamma_u, \gamma_{tu} \rangle}{|\gamma_u|} du = \int \langle T, \gamma_{ts} \rangle ds \\
&= \int \langle T, ((k - \alpha)N)_s \rangle ds = - \int k(k - \alpha) ds \\
&= - \int k^2 ds + \alpha \int k ds = 2\pi\alpha - \int k^2 ds = 0.
\end{aligned}$$

□

Another important fact for us is the following

Lemma 9. *The area of the disk bounded by the curve $\gamma(t)$ is increasing. That is,*

$$\frac{d}{dt}A(t) = \alpha L - 2\pi \geq 0.$$

Proof. Since

$$-2A(t) = \int \langle \gamma, N \rangle ds,$$

we have

$$\begin{aligned}
-2\frac{d}{dt}A(t) &= \int \frac{d}{dt} \langle \gamma, N \rangle ds + \int \langle \gamma, N \rangle \frac{d}{dt} ds \\
&= \int \langle (k - \alpha)N, N \rangle ds + \int \langle \gamma, -\partial_s(k - \alpha)T \rangle ds \\
&\quad + \int \langle \gamma, N \rangle (-k(k - \alpha)) ds \\
&= \int (k - \alpha) ds - \int \partial_s(k - \alpha) \langle \gamma, T \rangle ds - \int \langle \gamma, N \rangle k(k - \alpha) ds \\
&= \int (k - \alpha) ds + \int (k - \alpha) (\langle T, T \rangle + k \langle \gamma, N \rangle) ds \\
&\quad - \int \langle \gamma, N \rangle k(k - \alpha) ds \\
&= 2 \int (k - \alpha) ds = 2(2\pi - L\alpha)
\end{aligned}$$

Since

$$2\pi\alpha = \int k^2 ds \geq \frac{1}{L} \left(\int k ds \right)^2 = \frac{4\pi^2}{L}.$$

So

$$2\pi - \alpha L \leq 0,$$

which implies the result wanted. □

We remark that the above formula takes the equality when k is a constant, i.e. the curve is a circle.

We now consider the growth of the support function

$$P := \langle X, N \rangle$$

and the growth of α .

Lemma 10.

$$\begin{aligned}\partial_t P &= \partial_s^2 P + 2k - \alpha + k^2 P. \\ \partial_t \alpha &= -\frac{1}{\pi} \int (k_s)^2 ds + \frac{1}{2\pi} \int k^3 (k - \alpha) ds. \\ \partial_t \left(\frac{P}{k}\right) &= \partial_s^2 \left(\frac{P}{k}\right) + \left(P - \frac{1}{k}\right) \alpha + 2 \frac{k_s}{k} \partial_s \left(\frac{P}{k}\right).\end{aligned}$$

Proof.

$$\begin{aligned}\partial_t P &= \partial_t \langle \gamma, N \rangle = \langle \partial_t \gamma, N \rangle + \langle \gamma, \partial_t N \rangle \\ &= k - \alpha - k_s \langle \gamma, T \rangle.\end{aligned}$$

$$\partial_s P = \partial_s \langle \gamma, N \rangle = -k \langle \gamma, T \rangle.$$

$$\partial_s^2 P = -k_s \langle \gamma, T \rangle - k - k^2 \langle \gamma, N \rangle.$$

Combine the above computation, we have

$$\partial_t P = \partial_s^2 P + 2k - \alpha + k^2 P.$$

$$\partial_t \left(\frac{k}{P}\right) = \frac{k_t P - k P_t}{P^2}.$$

$$\partial_s \left(\frac{k}{P}\right) = \frac{k_s P - k P_s}{P^2}.$$

$$\partial_s^2 \left(\frac{k}{P}\right) = \frac{k_{ss} P - k P_{ss}}{P^2} - 2 \frac{P_s}{P} \partial_s \left(\frac{k}{P}\right).$$

Combine the above computation, we have

$$\begin{aligned}\partial_t \left(\frac{k}{P}\right) - \partial_s^2 \left(\frac{k}{P}\right) &= \frac{(k_t - k_{ss})P - k(P_t - P_{ss})}{P^2} + 2 \frac{P_s}{P} \partial_s \left(\frac{k}{P}\right) \\ &= \frac{k^2(k - \alpha)P - k(2k - \alpha + k^2 P)}{P^2} + 2 \frac{P_s}{P} \partial_s \left(\frac{k}{P}\right) \\ &= \frac{\alpha(1 - kP) - 2k^2}{P^2} + 2 \frac{P_s}{P} \partial_s \left(\frac{k}{P}\right).\end{aligned}$$

$$\begin{aligned}\partial_t \alpha &= \frac{1}{\pi} \int k k_t ds - \frac{1}{2\pi} \int k^3 (k - \alpha) ds \\ &= \frac{1}{\pi} \int k(k_{ss} + k^2(k - \alpha)) ds - \frac{1}{2\pi} \int k^3 (k - \alpha) ds \\ &= -\frac{1}{\pi} \int (k_s)^2 ds + \frac{1}{2\pi} \int k^3 (k - \alpha) ds.\end{aligned}$$

□

In principle we may consider the evolution of $f := \frac{k}{P-\rho}$ for ρ is a circle inside the region bounded by $\gamma(t)$ and get control of the curvature of the evolving curve. However, unlike the shrinking flow where $\alpha = 0$, we can not use the maximum principle to get the bound of f to show the convergence of the flow. So we need to use entropy estimate and integral estimate as done by Gage and Hamilton for the shrinking curve flow [7]. These will be done in below.

3. LONG TIME EXISTENCE

In this section, we derive key estimates of the curvature k of the evolving curve $\gamma(t)$.

In the following, we work with the general curve flow

$$(3.1) \quad \frac{\partial}{\partial t} \gamma = (k - \alpha)N + \eta T$$

where η will be determined later.

Similar to above, we have basic formulae

Lemma 11.

$$\partial_t \partial_s - \partial_s \partial_t = k(k - \alpha) \partial_s - \eta_s \partial_s.$$

$$\partial_t T = (\partial_s(k - \alpha) + k\eta)N.$$

$$\partial_t \theta = \partial_s k + k\eta.$$

$$\partial_t L = 0.$$

$$\partial_t A = \int (\alpha - k) ds.$$

We take η such that $\partial_t \theta = 0$, i.e. $\eta = -\partial_\theta k$.

Remark 12. Under this flow, the length is also preserved, and the variation of area is the same as $\eta = 0$.

Lemma 13.

$$\frac{\partial k}{\partial t} = k^2 \partial_\theta^2 k + k^2(k - \alpha).$$

Proof.

$$\begin{aligned} \partial_t k &= \partial_t \partial_s \theta = \partial_s \partial_t \theta + k(k - \alpha) \partial_s \theta - \partial_s \eta k \\ &= k^2 \partial_\theta^2 k + k^2(k - \alpha). \end{aligned}$$

□

Theorem 14. *Convexity is preserved along the flow (3.1). In fact, for any finite time $T < \infty$, such that the curve flow exists on $[0, T]$, we have $k(t)$ is uniformly bounded from below on $t \in [0, T]$*

Proof. By direct computation, we have

$$\partial_t \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \right) = k^2 \partial_\theta^2 \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \right) - 2k^3 \left(\partial_\theta \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \right) \right)^2 - k.$$

Before $\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}$ blow up, we have $k > 0$.

Since the curve is compact, we can take the maximum of $\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}$ at $(x_0, t_0) \in \gamma \times [0, T]$, where the curve flow exists on $[0, T]$. If $t_0 > 0$, we have

$$\frac{\partial}{\partial t} \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \right) \Big|_{(x_0, t_0)} \geq 0.$$

But

$$k^2 \partial_\theta^2 \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \right) \Big|_{x_0, t_0} \leq 0, \quad 2k^3 \left(\partial_\theta \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \right) \right)^2 \Big|_{x_0, t_0} = 0, \quad -k < 0$$

A contradiction. So

$$\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \leq \max(\frac{1}{k(0)}) - \frac{A(0)}{L}$$

That is $k(t) \geq \frac{1}{\frac{1}{C_1+C_2t}} = c(t)$, where C_1 and C_2 are positive constant. So the convexity is preserved along the flow. \square

Theorem 15. $\int \log k(\theta, t) d\theta$ is non increasing along the flow, So there is a uniform bound of $\int \log k(\theta, t) d\theta$ along the flow.

Proof.

$$\begin{aligned} \frac{\partial}{\partial t} \int \log k(\theta, t) d\theta &= \int k \partial_\theta^2 k + k(k - \alpha) d\theta \\ &= \int_0^{2\pi} -(\partial_\theta k)^2 + (k - \alpha)^2 d\theta + \alpha \int_0^{2\pi} (k - \alpha) d\theta. \end{aligned}$$

By definition, we have $\int_0^{2\pi} (k - \alpha) d\theta = 0$. Using the Wirtinger inequality, we have

$$\frac{\partial}{\partial t} \int_0^{2\pi} \log k(\theta, t) d\theta \leq 0.$$

Hence,

$$\int_0^{2\pi} \log k(\theta, t) d\theta \leq \int_0^{2\pi} \log k(\theta, 0) d\theta$$

\square

Theorem 16. [7] Suppose the curve flow exists on $[0, T)$. For any $\delta > 0$, we can find a constant $C(T)$ such that $k(\theta, t) \leq C(T)$ except on intervals of length less than or equal to δ .

Proof. If $k \geq C(t)$ on $a \leq \theta \leq b$ and $b - a \geq \delta$, then

$$\begin{aligned} \int_0^{2\pi} \log k(\theta, t) d\theta &\geq \delta \log C(T) + (2\pi - \delta) \log k_{\min}(t) \\ &\geq \delta \log C(T) + (2\pi - \delta) \log c(T) \end{aligned}$$

where $c(T)$ is the low bound of $k(t)$ on $[0, T)$. Since $\int_0^{2\pi} \log k(\theta, t) d\theta$ is non increasing, $C(T)$ is bounded above. \square

Lemma 17. We have

$$\int (\frac{\partial k}{\partial \theta})^2 d\theta \leq \int k^2 + D$$

where D is a constant which depends only on $\gamma(0)$.

Proof.

$$\begin{aligned} &\frac{\partial}{\partial t} \int (k - \alpha)^2 - (\frac{\partial k}{\partial \theta})^2 d\theta \\ &= 2 \int (k - \alpha)(\partial_t k - \partial_t \alpha) - 2 \frac{\partial k}{\partial \theta} \frac{\partial^2 k}{\partial \theta \partial t} d\theta \\ &= 2 \int (k - \alpha + \partial_\theta^2 k) \partial_t k d\theta - 2 \int (k - \alpha) \partial_t \alpha d\theta \\ &= 2 \int (k - \alpha + \partial_\theta^2 k)^2 k^2 d\theta + 2 \partial_t \alpha (2\pi \alpha - \int k d\theta) \end{aligned}$$

But

$$\int k d\theta = \int k^2 ds = 2\pi\alpha.$$

So

$$\frac{\partial}{\partial t} \int (k - \alpha)^2 - \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta = 2 \int (k - \alpha + \partial_\theta^2 k)^2 k^2 d\theta$$

Integrating the above inequality, we have

$$\int (k(t) - \alpha(t))^2 - \left(\frac{\partial k(t)}{\partial \theta}\right)^2 d\theta \geq \int (k(0) - \alpha(0))^2 - \left(\frac{\partial k(0)}{\partial \theta}\right)^2 d\theta = -D.$$

Then we have

$$\int \left(\frac{\partial k(t)}{\partial \theta}\right)^2 d\theta \leq \int (k(t) - \alpha(t))^2 + D \leq \int k^2 d\theta + D.$$

□

Theorem 18. *If $\int_0^{2\pi} \log k(\theta, t) d\theta$ is bounded on $[0, T)$, then $k(\theta, t)$ is uniformly bounded on $\gamma \times [0, T)$.*

Proof. For any given δ , by the above estimate, we have $k \leq C(T)$ except on intervals $[a, b]$ of length less than δ . On such an interval

$$k(\phi) = k(a) + \int_a^\phi \frac{\partial k}{\partial \theta} d\theta \leq C(T) + \sqrt{\delta} \left(\int_a^\phi \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta \right)^{1/2} \leq C(T) + \sqrt{\delta} \left(\int_a^\phi k^2 d\theta + D \right)^{1/2}.$$

This shows that if k_{max} is the maximum value of k , then

$$k_{max} \leq C(T) + \sqrt{\delta} (2\pi k_{max}^2 + D)^{1/2}.$$

By choosing δ small, we have

$$k_{max}^2 \leq \frac{2C^2(T) + 2\delta D}{1 - 4\pi\delta} \leq 4C^2(T).$$

□

Lemma 19. *If k is bounded, then $\frac{\partial k}{\partial \theta}$ is bounded.*

Proof.

$$\partial_t \partial_\theta k = k^2 \partial_\theta^3 k + 2k \partial_\theta k \partial_\theta^2 k + 3k^2 \partial_\theta k - 2\alpha k \partial_\theta k.$$

Since k is bounded, α is bounded. So $\partial_\theta k$ grows at most exponentially. □

In the following, we will use k' , etc, to denote the derivatives of $\gamma(t)$ w.r.t variable θ , $\partial_\theta k$, etc.

Lemma 20. *If k and k' are bounded, then $\int_0^{2\pi} (k'')^4 d\theta$ is bounded.*

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t} \int_0^{2\pi} (k'')^4 d\theta &= 4 \int_0^{2\pi} (k'')^3 (k^2 k'' + k^2 (k - \alpha))'' d\theta \\
&= -12 \int_0^{2\pi} (k'')^2 (k''') (k^2 k''' + 2k k' k'' + 3k^2 k' - 2\alpha k k') d\theta \\
&= -12 \int_0^{2\pi} k^2 (k'')^2 (k''')^2 + 2k k' (k'')^3 k''' + 3k^2 k' (k'')^2 k''' d\theta \\
&\quad + 24\alpha \int_0^{2\pi} k k' (k'')^2 k''' d\theta \\
&\leq C_1 \int_0^{2\pi} (k'')^4 (k')^2 d\theta + C_2 \int_0^{2\pi} k^2 (k')^2 (k'')^2 d\theta \\
&\quad + 24\alpha C_3 \int_0^{2\pi} (k')^2 (k'')^2 d\theta
\end{aligned}$$

By the bound of k, k' , we see that $\int_0^{2\pi} (k'')^4 d\theta$ grows at most exponentially. \square

Lemma 21. *If k, k' and $\int_0^{2\pi} (k'')^4 d\theta$ are bounded, so is $\int_0^{2\pi} (k''')^2 d\theta$.*

Proof.

$$\begin{aligned}
&\frac{\partial}{\partial \theta} \int_0^{2\pi} (k''')^2 d\theta \\
&= -2 \int_0^{2\pi} k'''' (k^2 k'' + k^3 - \alpha k^2)'' d\theta \\
&= -2 \int_0^{2\pi} k^2 (k''''')^2 + 4k k' k'''' k'''' + 2k (k'')^2 k'''' \\
&\quad + 2(k')^2 k'' k'''' + 3k^2 k'' k'''' + 6k (k')^2 k'''' \\
&\quad - 2\alpha (k')^2 k'''' - 2\alpha k k'' k'''' d\theta \\
&\leq C_1 \int_0^{2\pi} (k')^2 (k''')^2 d\theta + C_2 \int_0^{2\pi} (k'')^4 d\theta + C_3 \int_0^{2\pi} \frac{(k')^4}{k^2} (k'')^2 d\theta \\
&\quad + C_4 \int_0^{2\pi} k^2 (k'')^2 d\theta + C_5 \int_0^{2\pi} (k')^4 d\theta + C_6 \int_0^{2\pi} \frac{(k')^4}{k^2} d\theta + C_7 \int_0^{2\pi} (k'')^2 d\theta.
\end{aligned}$$

By the bounds of $k, k', \|k''\|_4$, we see that $\int_0^{2\pi} (k''')^2 d\theta$ grows at most exponentially. \square

Corollary 22. *Under the same hypothesis, k'' is bounded.*

Proof. In one dimension

$$\max |f|^2 \leq C \int |f'|^2 + f^2$$

We apply this to k'' . \square

Lemma 23. *If k, k' , and k'' are uniformly bounded, then so are k''' and all the higher derivatives.*

Proof. We compute

$$\begin{aligned}\frac{\partial}{\partial t}k''' &= (k^2k'' + k^3 - k^2\alpha)''' \\ &= k^2k^v + 6kk'k^{iv} + (8kk'' + 6k'^2 + 3k^2 - 2\alpha k)k''' \\ &\quad + (6k'(k'')^2 + 18kk'k'' + 6(k')^3 - 6\alpha k'k'')\end{aligned}$$

Since k, k', k'', α is bounded, k''' grows at most exponentially. Similarly, we can show that $k^{(n)}$ is bounded on finite intervals. \square

From the above analysis, we have

Theorem 24. *The curve flow does not blow up in finite time.*

Theorem 25. [5] *For the closed, convex C^2 curve $\gamma(t)$ in the plane, we have*

$$\pi \frac{L}{A} \leq \int_0^L k^2 ds,$$

where L, A and k are the length of the curve, the area it encloses, and its curvature respectively.

Theorem 26. *If the convex curve $\gamma(t)$ evolves according to (3.1), then the isoperimetric deficit $L^2 - 4\pi A$ is decreasing during the evolution process (1.1) and converges to zero as the time t goes to infinity.*

Proof. Since the length of the curve is preserved,

$$\frac{d}{dt}(L^2 - 4\pi A) = -4\pi \frac{d}{dt}A(t) = -4\pi(L\alpha - 2\pi) \leq 0.$$

From the above theorem 25, we have

$$\frac{d}{dt}(L^2 - 4\pi A) \leq -4\pi\left(\frac{L^2}{2A} - 2\pi\right) = -\frac{2\pi}{A}(L^2 - 4\pi A),$$

Since for any plane curve,

$$\frac{L^2}{4\pi} \geq A,$$

we have

$$\frac{d}{dt}(L^2 - 4\pi A) \leq -\frac{8\pi^2}{L^2}(L^2 - 4\pi A)$$

So

$$L^2 - 4\pi A(t) \leq C \exp\left(-\frac{8\pi^2}{L^2}t\right).$$

As $t \rightarrow \infty$, we have

$$L^2 - 4\pi A \rightarrow 0.$$

By Bonnesen inequality (see [14]), $\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{out} - r_{in})^2$, we have $r_{out} - r_{in} \rightarrow 0$, as $t \rightarrow \infty$. So the curve converge to a circle in Hausdorff sense. \square

4. C^∞ CONVERGENCE

We will follow Hamilton's estimate to prove the C^∞ convergence.

Define

$$k_w^* = \sup\{b \mid k(\theta) > b \text{ on some interval of length } w\}.$$

Lemma 27. [7]

$$k_w^*(t)r_{in}(t) \leq \frac{1}{1 - K(w)\left(\frac{r_{out}}{r_{in}} - 1\right)},$$

where r_{in} and r_{out} are the radii of the largest inscribed circle and the smallest circumscribed circle of the curve defined by the $k(\cdot, t)$ respectively, and K is a positive decreasing function of w with $K(0) = \infty$ and $K(\pi) = 0$.

Remark 28. Here we need the explicit formula of $K(w)$.

$$K(w) = \frac{2\cos(\frac{w}{2})}{1 - \cos(\frac{w}{2})}.$$

See also [7].

By the above section, we have the curve $\gamma(t)$ converges to a circle in Hausdorff sense as $t \rightarrow \infty$, i.e.

$$\pi(r_{out} - r_{in})^2 \leq L^2 - 4\pi A \rightarrow 0.$$

We also have $\pi r_{out}^2 \geq A(t) \geq A(0)$, and $r_{out} \rightarrow r_{in}$, So There is a sufficient large time T_1 such that $r_{in}(t) \geq \sqrt{\frac{A_0}{2\pi}}$ for $t \geq T_1$.

We first fix a small w . Then there is a sufficient large time $T_2 \geq T_1$, such that

$$K(w)\left(\frac{r_{out}}{r_{in}} - 1\right) \leq 1/2.$$

for $t \geq T_2$. By the theorem 27, we have $k_w^*(t)r_{in}(t) \leq 2$, i.e. $k_w^*(t) \leq 2\sqrt{\frac{2\pi}{A_0}}$ for $t \geq T_2$.

Then by a use of the method used by Gage and Hamilton in [7], we have

Theorem 29. *Curvature $k(t)$ uniformly bounded along the flow.*

Proof. We just need to consider the curvature k of the curve $\gamma(t)$ for $t \geq T_2$. First we fix a small $w < (\frac{1}{4\pi})^2$. Suppose that $[a, b]$ is a interval such that $k \geq k_w^*$. By condition, we have $|b - a| \leq w$ and $k(a) = k_w^*$. For any $\phi \in [a, b]$, we have

$$\begin{aligned} k(\phi) &= k(a) + \int_a^\phi \frac{\partial k}{\partial \theta} d\theta \leq k_w^* + \sqrt{w} \left(\int_a^\phi \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta \right)^{1/2} \\ &\leq k_w^* + \sqrt{w} \left(\int k^2 d\theta + D \right)^{1/2}. \end{aligned}$$

If k_{max} is the maximum value of k , then

$$k_{max} \leq k_w^* + \sqrt{w}(2\pi k_{max}^2 + D)^{1/2} \leq k_w^* + 2\pi\sqrt{w}k_{max} + \sqrt{w}D.$$

So for $t \geq T_2$, we have k is bounded uniformly. \square

Since $k(\theta, t)$ is uniformly bounded, $\int (\partial_\theta k)^2 d\theta$ is also uniformly bounded. Then we have $k(\theta, t)$ is equi-continuous. So for any sequence $k(\theta, t_i)$, we can choose a sequence $k(\theta, t_{i_n})$ converge uniformly to $k(\theta, \infty)$. But we know the curve converge to circle in the Hausdorff sense. So $k(\theta, \infty) = \text{const}$. Since every subsequences converge to $k(\theta, \infty) = \text{const}$, we have $k(\theta, t)$ converge to $k(\theta, \infty) = \text{const}$ uniformly.

Similar to [7], we have

Lemma 30. $\|k'\|_4$ are bounded by constants independent of t .

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t} \int (k')^4 d\theta &= 4 \int (k')^3 (k^2 k'' + k^2 (k - \alpha))' d\theta \\
&= -12 \int_0^{2\pi} k^2 (k')^2 (k'')^2 - 12 \int_0^{2\pi} (k')^2 k'' k^3 d\theta \\
&\quad - 8\alpha \int_0^{2\pi} (k')^4 k d\theta \\
&\leq 3 \int_0^{2\pi} k^4 (k')^2 d\theta - 8\alpha \int_0^{2\pi} (k')^4 d\theta
\end{aligned}$$

Since k converges to a constant, $\alpha(t)$ is converging to a constant. Using the Holder inequality, we have

$$\frac{\partial f}{\partial t} \leq C_1 f^{1/2} - C_2 f$$

where $f = \int_0^{2\pi} (k')^4 d\theta$, C_1 and C_2 is independent of t . By the lemma 5.7.4 of [7], we have $\|k'\|_4(t)$ is uniformly bounded. \square

Lemma 31. $\|k''\|_2$ is bounded by a constant which is independent of t .

Proof.

$$\begin{aligned}
&\partial_t \int_0^{2\pi} (k'')^2 d\theta \\
&= \int_0^{2\pi} k'' (\partial_t k'') d\theta \\
&= - \int_0^{2\pi} k''' (k^2 k'')' d\theta + \int_0^{2\pi} k'' ((k^3)'' - \alpha (k^2)'') d\theta \\
&= - \int_0^{2\pi} k^2 (k''')^2 d\theta - 2 \int_0^{2\pi} k k' k'' k''' d\theta \\
&\quad - 3 \int_0^{2\pi} k^2 k' k''' d\theta - 2\alpha \int_0^{2\pi} k (k'')^2 d\theta - 2\alpha \int_0^{2\pi} k'' (k')^2 d\theta
\end{aligned}$$

By the Cauchy inequality, we have

$$\begin{aligned}
\partial_t \int_0^{2\pi} (k'')^2 d\theta &\leq C_1 \int_0^{2\pi} (k' k'')^2 d\theta + C_2 \int_0^{2\pi} k^2 (k')^2 d\theta - 2\alpha \int_0^{2\pi} k (k'')^2 d\theta \\
&\quad - 2\alpha \int_0^{2\pi} k'' (k')^2 d\theta
\end{aligned}$$

We can control the first term by the inequality in above lemma 30,

$$\begin{aligned}
\partial_t \int_0^{2\pi} (k'')^2 d\theta &\leq C_2 \int_0^{2\pi} k^2 (k')^2 d\theta - 2\alpha \int_0^{2\pi} k (k'')^2 d\theta \\
&\quad - 2\alpha \int_0^{2\pi} k'' (k')^2 d\theta - C_3 \partial_t \int_0^{2\pi} (k')^4 d\theta \\
&\quad - C_4 \int_0^{2\pi} (k')^2 k'' k^3 d\theta - C_5 \int_0^{2\pi} (k')^4 d\theta \\
&\leq C_6 - C_7 \int_0^{2\pi} (k'')^2 d\theta - C_3 \partial_t \int_0^{2\pi} (k')^4 d\theta
\end{aligned}$$

We denote $\int_0^{2\pi} (k'')^2 d\theta$ by $f(t)$. Then we have

$$\partial_t f \leq C_6 - C_7 f - C_3 \partial_t \int_0^{2\pi} (k')^4 d\theta$$

Multiplying $e^{C_7 t}$ on both side and integrating, we have

$$e^{C_7 t} f(t) \Big|_0^t \leq C_6 \int_0^t e^{C_7 t} dt - C_3 \int_0^t e^{C_7 t} \partial_t \int_0^{2\pi} (k')^4 d\theta dt$$

So we have

$$\begin{aligned}
e^{C_7 t} f(t) &\leq \frac{C_6}{C_7} e^{C_7 t} + C_3 C_7 \int_0^t e^{C_7 t} \int_0^{2\pi} (k')^4 d\theta + C_8 \\
&\leq \frac{C_6}{C_7} e^{C_7 t} + C_3 M e^{C_7 t} + C_8.
\end{aligned}$$

where M is the bound of $\int_0^{2\pi} (k')^4 d\theta$ which is independent of t . So $\int_0^{2\pi} (k'')^2 d\theta$ is uniformly bounded. \square

As to lemma 5.7.8 of [7], we have

Lemma 32. $\|k'\|_\infty$ converges to 0 as $t \rightarrow \infty$.

Lemma 33. [7] For any $0 < \beta < 1$ we can choose A so that for $t > A$

$$\int (k'')^2 d\theta \geq 4\beta \int (k')^2 d\theta.$$

Lemma 34. For any $0 < \beta < 1$, there is a constant C_1 such that $\|k'\|_2 \leq C_1 e^{-2\beta C^2 t}$, where $k \rightarrow C$, as $t \rightarrow \infty$.

Proof.

$$\begin{aligned}
\partial_t \int (k')^2 d\theta &= 2 \int k' (k^2 k'' + k^3 - \alpha k^2)' d\theta \\
&= -2 \int k^2 (k'')^2 d\theta + 6 \int k^2 (k')^2 d\theta - 4\alpha \int k (k')^2 d\theta.
\end{aligned}$$

Since $k \rightarrow C$, as $t \rightarrow \infty$, we have

$$\partial_t \int (k')^2 d\theta \leq -2\beta C^2 \int (k')^2 d\theta.$$

So we complete the lemma. \square

Lemma 35. *We can find a constant C such that $\|k''\|_2 \leq Ce^{-2\beta t}$, where $\beta > 0$ is a uniform constant.*

Proof. By direct computation, we have

$$\begin{aligned}
& \partial_t \int (k'')^2 d\theta \\
&= 2 \int k'' (k_t)'' d\theta = 2 \int k'' (k^2 k'' + k^2 (k - \alpha))'' d\theta \\
&= -2 \int k^2 (k''')^2 d\theta - 4 \int k k' k'' k''' d\theta - 6 \int k^2 k' k''' d\theta \\
&\quad - 4\alpha \int (k')^2 k'' d\theta - 4\alpha \int k (k'')^2 d\theta \\
&\leq -2 \int k^2 (k''')^2 d\theta + 4\epsilon \int k^2 (k''')^2 d\theta + 1/\epsilon \int (k')^2 (k'')^2 d\theta \\
&\quad + 6(\epsilon \int (k k''')^2 + 1/4\epsilon \int k^2 (k')^2) - 4\alpha \int (k')^2 k'' d\theta \\
&\quad - 4\alpha \int k (k'')^2 d\theta
\end{aligned}$$

First we choose $\epsilon > 0$ small such that

$$\begin{aligned}
\partial_t \int (k'')^2 d\theta &\leq C_1 \int (k')^2 (k'')^2 d\theta + C_2 \int k^2 (k')^2 d\theta \\
&\quad - 4\alpha \int (k')^2 k'' d\theta - 4\alpha \int k (k'')^2 d\theta
\end{aligned}$$

Since $\|k'\|_\infty$ converges to 0, $k \rightarrow C$, $\alpha \rightarrow C$ as $t \rightarrow \infty$, we have

$$\partial_t \int (k'')^2 d\theta \leq -2C^2 \int (k'')^2 d\theta + C_2 C e^{-C_3 t}.$$

We denote $\int (k'')^2 d\theta$ by f . Then we have

$$\partial_t f \leq -2C^2 f + C_2 C e^{-C_3 t}.$$

By the lemma 5.7.6 of [7], we complete the proof. □

Lemma 36. *We can find uniform constants C and $\beta > 0$ such that $\|k''\|_4 \leq Ce^{-\beta t}$.*

Proof.

$$\begin{aligned}
& \partial_t \int (k'')^4 d\theta \\
&= 4 \int (k'')^3 (k^2 k'' + k^2 (k - \alpha))'' d\theta \\
&= -12 \int k^2 (k'')^2 (k''')^2 d\theta - 24 \int k k' (k'')^3 k''' d\theta \\
&\quad - 36 \int k^2 k' (k'')^2 k''' d\theta - 8\alpha \int (k')^2 (k'')^3 d\theta - 8\alpha \int k (k'')^4 d\theta \\
&\leq -12 \int k^2 (k'')^2 (k''')^2 d\theta + 24(\epsilon \int k^2 (k'')^2 (k''')^2 + 1/4\epsilon \int (k')^2 (k'')^4 d\theta) \\
&\quad + 36(\epsilon \int k^2 (k'')^2 (k''')^2 + 1/4\epsilon \int k^2 (k')^2 (k'')^2 d\theta) \\
&\quad - 8\alpha \int (k')^2 (k'')^3 d\theta - 8\alpha \int k (k'')^4 d\theta \\
&\leq C_1 \int (k')^2 (k'')^4 d\theta + C_2 \int k^2 (k')^2 (k'')^2 d\theta \\
&\quad - 8\alpha \int (k')^2 (k'')^3 d\theta - 8\alpha \int k (k'')^4 d\theta.
\end{aligned}$$

By Young inequality, we have

$$\begin{aligned}
\int (k')^2 (k'')^3 d\theta &\leq \epsilon \int (k')^{4/3} (k'')^4 d\theta + C(\epsilon) \int (k')^4 d\theta, \\
\int k^2 (k')^2 (k'')^2 d\theta &\leq \epsilon \int (k'')^4 d\theta + C(\epsilon) \int k^4 (k')^4 d\theta.
\end{aligned}$$

Since $\|k'\|_\infty \rightarrow 0$ as $t \rightarrow \infty$, we have

$$\partial_t \int (k'')^4 d\theta \leq -C_1 \int (k'')^4 d\theta + C_2 \int (k')^4 d\theta.$$

Denote $\int (k'')^4 d\theta$ by f . We have

$$\partial_t f \leq -C_1 f + C_2 e^{-C_3 t}$$

By lemma 5.7.6 in [7], we complete the proof. \square

Lemma 37. *There is some constant C such that $\|k'''\|_2 \leq C e^{-2\beta t}$, where $\beta > 0$ is a uniform constant.*

Proof.

$$\begin{aligned}
\partial_t \int (k''')^2 d\theta &= -2 \int k^2 (k''')^2 d\theta - 8 \int k k' k''' k'''' d\theta \\
&\quad - 4 \int (k')^2 k'' k'''' d\theta - 4 \int k (k'')^2 k'''' d\theta - 6 \int k^2 k'' k'''' d\theta \\
&\quad - 12 \int k (k')^2 k'''' d\theta - 4\alpha \int k (k''')^2 d\theta - 12\alpha \int k' k'' k'''' d\theta.
\end{aligned}$$

By Young inequality, we have

$$\begin{aligned}
\partial_t \int (k''')^2 d\theta &\leq -2 \int k^2 (k''')^2 d\theta + \epsilon \int k^2 (k''')^2 d\theta + C_1(\epsilon) \int (k')^2 (k''')^2 d\theta \\
&\quad + C_2(\epsilon) \int \frac{(k')^4 (k'')^2}{k^2} d\theta + C_3(\epsilon) \int (k'')^4 d\theta \\
&\quad + C_4(\epsilon) \int (k')^4 d\theta + C_5(\epsilon) \int k^2 (k'')^2 d\theta - 4\alpha \int k (k''')^2 d\theta \\
&\quad - 12\alpha(\epsilon) \int (k''')^2 d\theta + C_6(\epsilon) \int (k')^2 (k'')^2 d\theta.
\end{aligned}$$

By the above estimate of k' and k'' , for sufficiently large t , we have

$$\partial_t \int (k''')^2 d\theta \leq -C_6 \int (k''')^2 d\theta + C_7 e^{-\beta t}$$

The result follows from the lemma 5.7.5 in [7]. \square

In fact, the method in [7] can be applied to our case. The high order estimate of k is similar to [7], so we omit the detail.

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